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Abstract. The Baxter diagonalization of the XYZ hamiltonian is considered in the special case of the XY model. For this particular model the normalization properties of the basic Baxter states are obtained showing that the Baxter families form an overcomplete set of states. After choosing a simple subset of states which is complete, Bethe-type eigenstates of the XY hamiltonian are formed explicitly. By fixing a free parameter in Baxter's states an operator expression for these states is obtained. The XY hamiltonian is then diagonalized in terms of fermion operators closely related to the quasiparticle operators of the standard solution due to Lieb, Schultz, and Mattis. Finally relations are given expressing the Baxter states in terms of the standard quasiparticle states of this model.

1. Introduction

In a recent series of papers Baxter (1973) has obtained the eigenvectors and eigenvalues both of the transfer matrix for the eight-vertex model in lattice statistics and of the XYZ hamiltonian for a one-dimensional anisotropic Heisenberg chain of spins $\frac{1}{2}$. The XYZ hamiltonian may be written in the form

$$\mathscr{H}_{XYZ} = -\tfrac{1}{2}\sum_{j=1}^{N}\{(\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+) + \Gamma(\sigma_j^+\sigma_{j+1}^+ + \sigma_j^-\sigma_{j+1}^-) + \tfrac{1}{2}\Delta\sigma_j^z\sigma_{j+1}^z\}, \tag{1.1}$$

where $\sigma_j^{\pm}$, σ_j^z are the usual Pauli matrices with periodic boundary conditions $\sigma_{j+N} = \sigma_j$ and we assume the number N of spins to be even. Baxter shows that it is convenient to parametrize Γ and Δ in terms of Jacobi elliptic functions of modulus k as

$$\begin{aligned}\Gamma &= k\,\mathrm{sn}^2(2\eta, k),\\ \Delta &= \mathrm{cn}(2\eta, k)\,\mathrm{dn}(2\eta, k).\end{aligned} \tag{1.2}$$

In subsequent considerations we will work only in the regime defined by the condition $0 < k < 1$. If there exist integers L, m_1, and m_2 such that

$$L\eta = 2m_1K + \mathrm{i}m_2K', \tag{1.3}$$

(where K and K' are the complete elliptic integrals of the first kind of modulus k and complementary modulus $k' = (1-k^2)^{1/2}$ respectively) then Baxter (1973) gives a complete set of eigenvectors for $\mathscr{H}_{XYZ}$.

Baxter's work contains as special cases many previously solved models. It is of interest to examine Baxter's solution in these special cases not only to see if it sheds new light on these old problems but also to try to better understand in a simple case the rather complicated states Baxter has introduced in his diagonalization of $\mathscr{H}_{XYZ}$. In particular, the XY model together with several previously solved problems in lattice statistics (Baxter 1973) corresponds to taking $L = 4$ in (1.3). The simplest of several ways to obtain the XY model is to choose $L = 4$, $m_1 = 1$, $m_2 = 0$ (J G Valatin, private communication). This choice corresponds to

$$\eta = \tfrac{1}{2}K, \qquad \Gamma = k, \qquad \Delta = 0. \tag{1.4}$$

We then have the XY hamiltonian in the form

$$\begin{aligned} \mathscr{H}_{XY} &= -\tfrac{1}{2}\sum_{j=1}^{N} \mathscr{H}_{j,j+1}, \\ \mathscr{H}_{j,j+1} &= (\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+) + k(\sigma_j^+\sigma_{j+1}^+ + \sigma_j^-\sigma_{j+1}^-). \end{aligned} \tag{1.5}$$

There is a familiar diagonalization of $\mathscr{H}_{XY}$ in which it is transformed into the hamiltonian of a system of non-interacting fermions (Lieb *et al* 1961). However, Baxter's solution (Baxter 1973) is a Bethe-type solution (Bethe 1931) for the wavefunctions of $\mathscr{H}_{XY}$, and it is interesting to ask how the Baxter wavefunctions are related to the states of the operator diagonalization. We shall see below how Baxter's wavefunctions are constructed out of an overcomplete set of basis vectors in the 2^N dimensional space W in which $\mathscr{H}_{XY}$ acts. The overcompleteness gives great freedom in selecting a complete subset of states with which to diagonalize $\mathscr{H}_{XY}$. We show how a simple choice of Baxter states leads again to an operator diagonalization of $\mathscr{H}_{XY}$, but in terms of operators slightly different from the fermion operators of the standard solution (Lieb *et al* 1961). In obtaining this result we will also see how the excited states of the standard solution with many fermions present correspond to superpositions of the Baxter states with various numbers of Baxter's 'spins' flipped.

Thus in § 2 we introduce the Baxter families of states to be used as a basis in the spin space W and collect various simple properties of them. In § 3 we choose a subset of the Baxter families in order to construct eigenstates of $\mathscr{H}_{XY}$ by the Bethe ansatz method. In § 4 we find operator expressions for the states of the Baxter families. In § 5 we diagonalize $\mathscr{H}_{XY}$ in terms of fermion operators closely related both to the Baxter states and to the conventional quasiparticle operators (Lieb *et al* 1961). Finally in § 6 we give some relations between the Baxter states and the states of the usual quasiparticle solution.

2. Baxter families of vectors

Baxter (1973) has introduced some remarkable families of vectors in the 2^N dimensional space W of the XYZ model. Each vector in these families depends upon two free parameters s and t as well as upon the parameter v used by Baxter to parametrize the eight-vertex transfer matrix (Baxter 1972a). A suitable adjustment of s and t enables one to put these vectors in a form independent of v but still dependent upon s and t (Baxter 1973). As mentioned by Baxter (1973) the dependence upon s and t reflects a kind of degeneracy because as these parameters are varied each of Baxter's eigenvectors moves through a subspace of the 2^N dimensional space W.

In what follows it is much simpler to work with these Baxter vectors if we fix the parameter t relative to s but still leave s free. If we do this, we can introduce the Baxter families for the XY model in the following straightforward manner. Define

$$p(l, s) = \sqrt{k}\,\mathrm{sn}(s + lK, k), \tag{2.1}$$

where sn is one of the Jacobi elliptic functions (Whittaker and Watson 1965) of modulus k, l is an integer, and s is a free parameter. On each spin site j define an orthonormal pair of two-component spinors by

$$\begin{aligned} \varphi_{l_j,l_j+1} &= (1+p^2(l_j,s))^{-1/2}\begin{pmatrix} p(l_j,s) \\ 1 \end{pmatrix}, \\ \varphi_{l_j,l_j-1} &= (1+p^2(l_j,s))^{-1/2}\begin{pmatrix} 1 \\ -p(l_j,s) \end{pmatrix}. \end{aligned} \tag{2.2}$$

In Baxter's terminology $\varphi_{l,l+1}$ and $\varphi_{l,l-1}$ are respectively up arrows and down arrows (Baxter 1973). This pair of spinors is orthonormal only because we have fixed Baxter's parameter t. The fact that the spinors are orthonormal means that they do describe 'spins', but 'spins' that are up and down with respect to an axis rotated with respect to the z axis specified by the σ_j^z operator on site j. The parameter s is free, but it is convenient henceforth to assume it is real in order that the spinors (2.2) are real. Now define direct product states in the 2^N dimensional space W by (Baxter 1973)

$$\psi(l_1, l_2, \ldots, l_N, l_{N+1}) = \varphi_{l_1,l_2}\otimes\varphi_{l_2,l_3}\ldots\otimes\varphi_{l_{N-1},l_N}\otimes\varphi_{l_N,l_{N+1}}, \tag{2.3}$$

where the sequence of integers l_j is constrained to satisfy

$$l_{j+1} = l_j \pm 1, \qquad l_{N+1} = l_1 + rL, \tag{2.4}$$

with r an integer and $L = 4$. We shall see below that in fact the l_j are defined only *modulo* L, hence $l_{N+1} = l_1 + rL$ is a kind of periodic boundary condition. Such a condition is essential in order that each of Baxter's families of states should be closed under the action of $\mathscr{H}_{XY}$. In the state (2.3) we have an up or a down 'spin' on each site but with respect to an axis which rotates through differing angles as we go from site to site.

Denote by n the number of down 'spins' which occur in the product state ψ. Then we have that

$$l_{N+1} = l_1 + N - 2n, \tag{2.5}$$

and, because of (2.4), that

$$n = \tfrac{1}{2}N - 2r. \tag{2.6}$$

Thus, if $\frac{1}{2}N$ is even, n takes the even values $0, 2, 4, \ldots, N$, while, if $\frac{1}{2}N$ is odd, n takes odd values $1, 3, 5, \ldots, N-1$. Let x_i, $i = 1, 2, \ldots, n$ denote the sites of the chain where the n down 'spins' occur. Writing $l = l_1$ we may denote (2.3) by

$$\psi(l_1, l_2, \ldots, l_{N+1}) = \psi(l; x_1, x_2, \ldots, x_n), \tag{2.7}$$

where

$$1 \leqslant x_1 < x_2 < x_3 \ldots < x_n \leqslant N,$$

and

$$\begin{aligned} l_j &= l + j - 1 && \text{for } j \leqslant x_1, \\ l_j &= l + j - 1 - 2i && \text{for } x_i < j \leqslant x_{i+1}, \\ l_j &= l + j - 1 - 2n && \text{for } x_n < j. \end{aligned} \tag{2.8}$$

The simplest such state is the $n = 0$ state,

$$\psi(l) = \varphi_{l,l+1} \otimes \varphi_{l+1,l+2} \cdots \otimes \varphi_{l+N-1,l+N}. \tag{2.9}$$

It is a simple property of the sn function that $p(l+2, s) = -p(l, s)$. Hence there are only four distinct pairs of spinors (2.2) possible on each site corresponding to $l_j = 1, 2, 3, 4$. It immediately follows from (2.7) and (2.8) that

$$\psi(l+4; x_1, x_2, \ldots, x_n) = \psi(l; x_1, x_2, \ldots, x_n). \tag{2.10}$$

An important operator which commutes with $\mathscr{H}_{XY}$ is

$$U = \sigma_1^z \sigma_2^z \ldots \sigma_N^z. \tag{2.11}$$

Let us ask what effect U has on the Baxter states ψ introduced above. First note that on site j we have

$$\begin{aligned} \sigma_j^z \varphi_{l_j,l_j+1} &= -\varphi_{l_j+2,l_j+3}, \\ \sigma_j^z \varphi_{l_j,l_j-1} &= \varphi_{l_j+2,l_j+1}. \end{aligned} \tag{2.12}$$

From this result we see at once that

$$U\psi(l; x_1, x_2, \ldots, x_n) = (-1)^n \psi(l+2; x_1, x_2, \ldots; x_n). \tag{2.13}$$

Defining projection operators $P_\pm$ which commute with $\mathscr{H}_{XY}$ by

$$P_\pm = \tfrac{1}{2}(1 \pm U), \tag{2.14}$$

we see that

$$P_\pm \psi(l; x_1, \ldots, x_n) = \tfrac{1}{2}\{\psi(l; x_1, \ldots, x_n) \pm (-1)^n \psi(l+2; x_1, \ldots, x_n)\}. \tag{2.15}$$

The states ψ for l and $l+2$ have simple normalization properties. As shown in appendix 1 we have

$$(\psi(l; x_1, x_2, \ldots, x_n), \psi(l; y_1, y_2, \ldots, y_m)) = \delta_{n,m} \delta_{x_1,y_1} \delta_{y_2,y_2} \ldots \delta_{x_n,y_n}, \tag{2.16}$$

and

$$(\psi(l; x_1, x_2, \ldots, x_n), \psi(l+2; y_1, y_2, \ldots, y_m)) = \left(\frac{1-k}{1+k}\right)^{N/2} \delta_{n,m} \delta_{x_1,y_1} \delta_{x_2,y_2} \ldots \delta_{x_n,y_n}. \tag{2.17}$$

The scalar product between states for l and $l+1$ is not so simple.

From (2.16) with l fixed and n taking the values allowed by (2.6) we see that there are 2^{N-1} orthonormal vectors $\psi(l; x_1, x_2, \ldots, x_n)$. As $l = 1, 2, 3, 4$ we generate 2^{N+1} vectors, an overcomplete set. From (2.17) it is clear that so long as $k \neq 0$ we may choose the l odd states only ($l = 1, 3$) or the l even states only ($l = 2, 4$) in order to get a complete set of 2^N independent vectors in W. Of course there are many other ways to pick 2^N independent states out of the available 2^{N+1} states. But if we want to diagonalize both $\mathscr{H}_{XY}$ and $P_\pm$, then either the odd l or even l states are the obvious choice. This choice is convenient also because if $\mathscr{H}_{XY}$ is applied to $\psi(l; x_1, \ldots, x_n)$, then either the value of l is unchanged or shifted by two. Thus $\mathscr{H}_{XY}$ acting on the set of l odd (even) states gives back the set of l odd (even) states. One way to see this is to use Baxter's relation between $\mathscr{H}_{XYZ}$ and the logarithmic derivative of the eight-vertex transfer matrix (Baxter 1972b) together with Baxter's equation expressing the action of the transfer matrix on the state $\psi(l; x_1, \ldots, x_n)$ (Baxter 1973). A more direct way to see this is to note that in (1.5) we have expressed $\mathscr{H}_{XY}$ as a sum of operators $\mathscr{H}_{j,j+1}$ which act only on pairs of adjacent spin sites. In appendix 1 we show that the effect of $\mathscr{H}_{j,j+1}$ upon the pair of spinors at sites j and $j+1$ is given by the following set of equations:

$$\begin{aligned}\mathscr{H}_{j,j+1}&\varphi_{l_j,l_j+1}\otimes\varphi_{l_j+1,l_j+2}\\ &= A(l_j,s)\varphi_{l_j,l_j+1}\otimes\varphi_{l_j+1,l_j+2}+B(l_j,s)\varphi_{l_j,l_j-1}\otimes\varphi_{l_j+1,l_j+2}\\ &\quad -B(l_j+1,s)\varphi_{l_j,l_j+1}\otimes\varphi_{l_j+1,l_j},\end{aligned}\tag{2.18a}$$

$$\begin{aligned}\mathscr{H}_{j,j+1}&\varphi_{l_j,l_j+1}\otimes\varphi_{l_j+1,l_j}\\ &= -A(l_j,s)\varphi_{l_j,l_j+1}\otimes\varphi_{l_j+1,l_j}+C(l_j,s)\varphi_{l_j,l_j-1}\otimes\varphi_{l_j-1,l_j}\\ &\quad + B(l_j,s)\varphi_{l_j,l_j-1}\otimes\varphi_{l_j+1,l_j}-B(l_j+1,s)\varphi_{l_j,l_j+1}\otimes\varphi_{l_j+1,l_j+2},\end{aligned}\tag{2.18b}$$

$$\begin{aligned}\mathscr{H}_{j,j+1}&\varphi_{l_j,l_j-1}\otimes\varphi_{l_j-1,l_j}\\ &= -A(l_j-1,s)\varphi_{l_j,l_j-1}\otimes\varphi_{l_j-1,l_j}+C(l_j,s)\varphi_{l_j,l_j+1}\otimes\varphi_{l_j+1,l_j}\\ &\quad +B(l_j,s)\varphi_{l_j,l_j+1}\otimes\varphi_{l_j-1,l_j}-B(l_j-1,s)\varphi_{l_j,l_j-1}\otimes\varphi_{l_j-1,l_j-2},\end{aligned}\tag{2.18c}$$

$$\begin{aligned}\mathscr{H}_{j,j+1}&\varphi_{l_j,l_j-1}\otimes\varphi_{l_j-1,l_j-2}\\ &= A(l_j-1,s)\varphi_{l_j,l_j-1}\otimes\varphi_{l_j-1,l_j-2}+B(l_j,s)\varphi_{l_j,l_j+1}\otimes\varphi_{l_j-1,l_j-2}\\ &\quad -B(l_j-1,s)\varphi_{l_j,l_j-1}\otimes\varphi_{l_j-1,l_j}.\end{aligned}\tag{2.18d}$$

The coefficients $A(l,s)$, $B(l,s)$, and $C(l,s)$ are defined by

$$A(l,s) = k\,\mathrm{sn}(2s+2lK),\tag{2.19a}$$

$$B(l,s) = \sqrt{k}\,\mathrm{cn}(s+lK)\,\mathrm{dn}(s+lK)/(1+p^2(l,s)),\tag{2.19b}$$

$$C(l,s) = (1+k)(1-p^2(l,s))/(1+p^2(l,s)).\tag{2.19c}$$

Some contemplation shows that, after summing $\mathscr{H}_{j,j+1}$ over all sites to obtain the effect of $\mathscr{H}_{XY}$ upon $\psi(l;x_1,\ldots,x_n)$, the terms involving $B(l,s)$ cancel out leaving a result dependent upon the coefficients $A(l,s)$ and $C(l,s)$. From (2.18) one sees explicitly that when $\mathscr{H}_{XY}$ acts upon $\psi(l;x_1,\ldots,x_n)$ the number n of down 'spins' is conserved and the individual x_i may be changed to $x_i\pm 1$.

As an example, suppose $\frac{1}{2}N$ is odd so that there are Baxter states with $n = 1$. For such states one sees from (2.18) that

$$\mathscr{H}_{XY}\psi(l;x_1) = A(l+x_1,s)\psi(l;x_1)-\tfrac{1}{2}C(l+x_1,s)\psi(l;x_1-1)-\tfrac{1}{2}C(l+x_1+1,s)\psi(l;x_1+1),\tag{2.20}$$

where to include correctly the values $x_1 = 1, N$ on the left-hand side of this equation, we make the supplementary definitions

$$\psi(l;0) = \psi(l-2;N),\qquad \psi(l;N+1) = \psi(l+2;1).\tag{2.21}$$

If $\frac{1}{2}N$ is even, one has the very simple $n = 0$ states (2.9). From (2.18) and (2.19) one finds at once Baxter's (1973) result,

$$\mathscr{H}_{XY}\psi(l) = 0,\tag{2.22}$$

that is, that the $n = 0$ states in the XY case are all zero energy eigenstates. For the case $n = 2$ we may write

$$\begin{aligned}\mathscr{H}_{XY}\psi(l; x_1, x_2) &\\ = (A(l+x_1, s)+A(l+x_2, s))\psi(l; x_1, x_2)-\tfrac{1}{2}C(l+x_1, s)\psi(l; x_1-1, x_2)&\\ -\tfrac{1}{2}C(l+x_1+1, s)\psi(l; x_1+1, x_2)-\tfrac{1}{2}C(l+x_2, s)\psi(l; x_1, x_2-1)&\\ -\tfrac{1}{2}C(l+x_2+1, s)\psi(l; x_1, x_2+1).& \qquad (2.23)\end{aligned}$$

Again, to fit the cases $x_1 = 1$, $x_2 = N$, and $x_2 = x_1+1$ on the left-hand side of the equation we need the supplementary definitions

$$\begin{aligned}\psi(l; x_1, x_1) &= 0,\\ \psi(l; 0, x_2) &= \psi(l-2; x_2, N), \qquad (2.24)\\ \psi(l; x_1, N+1) &= \psi(l+2; 1, x_1).\end{aligned}$$

3. Bethe-type eigenstates for $n = 1, 2$

From these Baxter familes of vectors for each allowed value of n we can now form eigenvectors of $\mathscr{H}_{XY}$. This is essentially the same calculation which Baxter (1973) has done generally for the eight-vertex model transfer matrix, but it is much easier to follow if we do it specifically for $\mathscr{H}_{XY}$. We will carry out the calculation explicitly below in the cases $n = 1$ ($\frac{1}{2}N$ odd) and $n = 2$ ($\frac{1}{2}N$ even).

For simplicity, let us begin with the $n = 1$ ($\frac{1}{2}N$ odd) case. Since $P_\pm$ commutes with $\mathscr{H}_{XY}$ we may work in the two subspaces $W_\pm = P_\pm W$ separately by defining

$$\psi_\pm(l; x_1) = P_\pm\psi(l; x_1) = \tfrac{1}{2}(\psi(l; x_1)\mp\psi(l+2; x_1)). \qquad (3.1)$$

Because of the overcompleteness of the states ψ we may choose either $l = 1$ or $l = 2$ in (3.1). To be definite, let us choose $l = 1$ at this stage and later we can get the $l = 2$ result by merely shifting s to $s+K$. Remembering that

$$A(l+2, s) = A(l, s) \qquad \text{and} \qquad C(l+2, s) = C(l, s)$$

we get from (2.20)

$$\begin{aligned}\mathscr{H}_{XY}\psi_\pm(1; x_1) &\\ = A(x_1-1, s)\psi_\pm(1; x_1)-\tfrac{1}{2}C(x_1-1, s)\psi_\pm(1; x_1-1)-\tfrac{1}{2}C(x_1, s)\psi_\pm(1; x_1+1).& \qquad (3.2)\end{aligned}$$

Now try to find an eigenstate $\Psi^{(1)}_\pm$ of $\mathscr{H}_{XY}$ such that

$$\mathscr{H}_{XY}\Psi^{(1)}_\pm = E^{(1)}_\pm\Psi^{(1)}_\pm, \qquad (3.3)$$

$$\Psi^{(1)}_\pm = \sum_{x_1=1}^{N} g_\pm(x_1, s)\psi_\pm(1; x_1). \qquad (3.4)$$

If we put (3.4) into (3.3) and use (3.2), we may equate coefficients of $\psi_\pm(1, x_1)$ on either side of the equation to obtain, for $x_1 \neq 1, N$,

$$E^{(1)}_\pm g_\pm(x_1, s) = A(x_1-1, s)g_\pm(x_1, s)-\tfrac{1}{2}C(x_1, s)g_\pm(x_1+1, s)-\tfrac{1}{2}C(x_1-1, s)g_\pm(x_1-1, s). \qquad (3.5)$$

If we introduce the ratio of coefficients

$$f_\pm(x_1, s) = \frac{g_\pm(x_1+1, s)}{g_\pm(x_1, s)}, \tag{3.6}$$

(3.5) becomes

$$E_\pm^{(1)} = A(x_1-1, s) - \tfrac{1}{2}C(x_1, s)f_\pm(x_1, s) - \tfrac{1}{2}C(x_1-1, s)(1/f_\pm(x_1-1, s)). \tag{3.7}$$

This is now a kind of recurrence relation to be solved for $f_\pm(x_1, s)$ in terms of $E_\pm^{(1)}$. As shown directly in appendix 2, or by referring to Baxter's proof in general for the eight-vertex model (Baxter 1973), the solution of (3.7) is given by

$$f_\pm(x_1, s) = f(x_1, s, u) = \frac{\mathrm{i}G(u^\pm)}{G(u^\pm + \frac{1}{2}\mathrm{i}K' + s + x_1 K)}, \tag{3.8}$$

where the parameter $u^\pm$ is related to $E_\pm^{(1)}$ by

$$E_\pm^{(1)}(u^\pm) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}u^\pm}\ln G(u^\pm) = \frac{1}{\mathrm{sn}(2u^\pm, k)}, \tag{3.9}$$

and the function $G(u)$ is defined† by

$$G(u) = \frac{1}{k'}\frac{\mathrm{cn}(u, k)\,\mathrm{dn}(u, k)}{\mathrm{sn}(u, k)}, \tag{3.10a}$$

or

$$G(v-\eta) = \frac{H(v+\eta)\Theta(v+\eta)}{H(v-\eta)\Theta(v-\eta)}, \tag{3.10b}$$

where H and Θ are Jacobi theta functions, $\eta = \frac{1}{2}K$ as in (1.4), and v is Baxter's transfer matrix parameter, related to u by $u = v - \eta$. Following Baxter (1973) define wave-numbers $q^\pm$ by the relation

$$G(u^\pm) = G(v^\pm - \eta) = \frac{H(v^\pm+\eta)\Theta(v^\pm+\eta)}{H(v^\pm-\eta)\Theta(v^\pm-\eta)} = \exp(\mathrm{i}q^\pm). \tag{3.11}$$

Then a simple argument based on the theorem that there is an algebraic relation between any two elliptic functions with the same periods (Whittaker and Watson 1965) leads to

$$E_\pm^{(1)} = (\cos^2 q^\pm + k^2\sin^2 q^\pm)^{1/2}, \tag{3.12}$$

a form familiar from the usual quasiparticle solution. Because $G(u)$ is of order two (Whittaker and Watson 1965), for every value of $q^\pm$ in (3.11) there will be two possible values for $u^\pm$. As explained in appendix 2 we choose those $u^\pm$ which lead to the square root in (3.12) being always positive. If we now fix

$$g_\pm(1, s) = g(1, s, u^\pm) = G(u^\pm) = \exp(\mathrm{i}q^\pm), \tag{3.13}$$

then from (3.8) we have for x_1 odd,

$$g(x_1, s, u^\pm) = \exp(\mathrm{i}q^\pm x_1) \tag{3.14a}$$

and for x_1 even,

$$g(x_1, s, u^\pm) = -\mathrm{i}G(u^\pm + \tfrac{1}{2}\mathrm{i}K' + s)\exp(\mathrm{i}q^\pm x_1). \tag{3.14b}$$

† The function $G(u)$ was introduced in connection with the XY model by J G Valatin.

As yet, $u^{\pm}$ and the associated wavenumbers $q^{\pm}$ are undetermined parameters. The allowed values of $q^{\pm}$ are fixed by a boundary condition which follows from looking explicitly at the terms in (3.3) and (3.5) when $x_1 = 1, N$. From (2.21) we see that

$$\psi_{\pm}(1;0) = \mp\psi_{\pm}(1;N), \qquad \psi_{\pm}(1;N+1) = \mp\psi_{\pm}(1;1),$$

hence from the ends of the chain arise two equations,

$$E_{\pm}^{(1)}g(1,s,u^{\pm}) = A(0,s)g(1,s,u^{\pm}) - \tfrac{1}{2}C(1,s)g(2,s,u^{\pm}) \pm \tfrac{1}{2}C(N,s)g(N,s,u^{\pm}),$$

and

$$E_{\pm}^{(1)}g(N,s,u^{\pm}) = A(N-1,s)g(N,s,u^{\pm}) \pm \tfrac{1}{2}C(0,s)g(1,s,u^{\pm}) - \tfrac{1}{2}C(N-1,s)g(N-1,s,u^{\pm}).$$

We satisfy these as well as (3.5) if we impose the boundary condition

$$g(x_1+N,s,u^{\pm}) = \mp g(x_1,s,u^{\pm}), \tag{3.15}$$

equivalent to

$$\exp(iq^{\pm}N) = \mp 1. \tag{3.16}$$

Thus in each of the two subspaces $W_{\pm}$ we get the usual set of wavenumbers. In W_+ we have

$$q^{+} = \pm\frac{\pi}{N}, \pm\frac{3\pi}{N}, \pm\frac{5\pi}{N}, \ldots, \pm\frac{(N-1)\pi}{N}, \tag{3.17}$$

while in W_- we have

$$q^{-} = 0, \pm\frac{2\pi}{N}, \pm\frac{4\pi}{N}, \ldots, \pm\frac{(N-2)\pi}{N}, \pi. \tag{3.18}$$

Next let us consider the case that $\frac{1}{2}N$ is even so that there is a Baxter family of $n = 2$ states. Once again define

$$\psi_{\pm}(l;x_1,x_2) = P_{\pm}\psi(l;x_1,x_2) = \tfrac{1}{2}(\psi(l;x_1,x_2) \pm \psi(l+2;x_1,x_2)), \tag{3.19}$$

and, as above, choose $l = 1$. Then from (2.23) we have

$$\begin{aligned}\mathscr{H}_{XY}\psi_{\pm}(1;x_1,x_2) = {} & (A(x_1-1,s)+A(x_2-1,s))\psi_{\pm}(1;x_1,x_2) \\ & -\tfrac{1}{2}C(x_1-1,s)\psi_{\pm}(1;x_1-1,x_2) - \tfrac{1}{2}C(x_1,s)\psi_{\pm}(1;x_1+1,x_2) \\ & -\tfrac{1}{2}C(x_2-1,s)\psi_{\pm}(1;x_1,x_2-1) - \tfrac{1}{2}C(x_2,s)\psi_{\pm}(1;x_1,x_2+1).\end{aligned} \tag{3.20}$$

Again try to form an eigenstate $\Psi_{\pm}^{(2)}$ such that

$$\mathscr{H}_{XY}\Psi_{\pm}^{(2)} = E_{\pm}^{(2)}\Psi_{\pm}^{(2)}, \tag{3.21}$$

$$\Psi_{\pm}^{(2)} = \sum_{x_1<x_2} h_{\pm}(x_1,x_2)\psi_{\pm}(1;x_1,x_2). \tag{3.22}$$

Insert (3.22) into (3.21), use (3.20), and then equate coefficients of $\psi_{\pm}(1;x_1,x_2)$ in the case that $x_2 \neq x_1+1, x_1 \neq 1, x_2 \neq N$. We get

$$\begin{aligned}E_{\pm}^{(2)}h_{\pm}(x_1,x_2) = {} & (A(x_1-1,s)+A(x_2-1,s))h_{\pm}(x_1,x_2) - \tfrac{1}{2}C(x_1-1,s)h_{\pm}(x_1-1,x_2) \\ & -\tfrac{1}{2}C(x_1,s)h_{\pm}(x_1+1,x_2) - \tfrac{1}{2}C(x_2-1,s)h_{\pm}(x_1,x_2-1) \\ & -\tfrac{1}{2}C(x_2,s)h_{\pm}(x_1,x_2+1).\end{aligned} \tag{3.23}$$

If we try $h_{\pm}(x_1, x_2) = g(x_1, s, u_1^{\mp})g(x_2, s, u_2^{\mp})$, we find a solution of (3.23) with

$$E_{\pm}^{(2)} = \frac{1}{\mathrm{sn}(2u_1^{\mp})} + \frac{1}{\mathrm{sn}(2u_2^{\mp})}.$$

However, $h_{\pm}(x_1, x_2) = g(x_1, s, u_2^{\mp})g(x_2, s, u_1^{\mp})$ is also a solution of (3.23) with the same value of $E_{\pm}^{(2)}$. Therefore we may choose as a general solution of (3.23),

$$h_{\pm}(x_1, x_2) = c_{12}g(x_1, s, u_1^{\mp})g(x_2, s, u_2^{\mp}) + c_{21}g(x_1, s, u_2^{\mp})g(x_2, s, u_1^{\mp}),$$

where the constants c_{12} and c_{21} are chosen in the usual way (Bethe 1931) to ensure that (3.23) is valid also when $x_2 = x_1 + 1$. This gives the familiar consistency condition

$$h_{\pm}(x_1, x_1) + h_{\pm}(x_1 + 1, x_1 + 1) = 0, \tag{3.24}$$

which tells us that $c_{12} = -c_{21}$. Thus a solution valid even at $x_2 = x_1 + 1$ is

$$h_{\pm}(x_1, x_2) = g_{\pm}(x_1, s, u_1^{\mp})g_{\pm}(x_2, s, u_2^{\mp}) - g_{\pm}(x_1, s, u_2^{\mp})g_{\pm}(x_2, s, u_1^{\mp}). \tag{3.25}$$

To determine $u_1^{\mp}$ and $u_2^{\mp}$ we use boundary conditions arising from the terms involving $\psi_{\pm}(1; 1, x_2)$, $\psi_{\pm}(1; 1, N)$, and $\psi_{\pm}(1; x_1, N)$ in (3.21). One finds that

$$h_{\pm}(x_1 + N, x_2) = h_{\pm}(x_1, x_2 + N) = \mp h_{\pm}(x_1, x_2), \tag{3.26}$$

which leads to the same wavenumbers $q^{\pm}$ in each of the subspaces $W_{\pm}$ that we determined previously in (3.17), (3.18) for $n = 1$.

One could now proceed to construct eigenstates out of the other allowed Baxter families ($n = 4, 6, 8, \ldots, N$ for $\frac{1}{2}N$ even) in the usual Bethe (1931) manner. This procedure would give straightforward generalizations of (3.22) and (3.25) in which we would have a superposition of Baxter states, each with n down 'spins', with coefficients like (3.25) which would be given by determinants of the single particle functions $g(x, s, u^{\pm})$. In such fashion we could find a complete set of eigenstates for $\mathscr{H}_{XY}$. However, the fact that we find determinants of the single particle functions $g(x, s, u^{\pm})$ implies that the eigenstates are made up of fermion excitations and suggests that the Baxter solution is closely related to the familiar diagonalization of $\mathscr{H}_{XY}$ in terms of fermion quasiparticle operators.

It is worth emphasizing that this Bethe-type solution is carried out with the parameter s held fixed. If now we vary s, the energy eigenvalues do not change but the eigenvectors move in the 2^N dimensional space W. The parameter s thus is related to the existence of degeneracy among the eigenstates of $\mathscr{H}_{XY}$. At present we will not consider further this problem of degeneracy, but it should be possible to examine it in detail for the XY model.

4. Operator expression of the Baxter states ($\frac{1}{2}N$ even)

Thus far the parameter s has been undetermined apart from requiring it to be real. When we illustrated the Bethe-type eigenstates above, we used states $\psi(l; x_1, \ldots, x_n)$ with $l = 1, 3$ as the basic set. The results of § 3 also contain the choice $l = 2, 4$ if we shift s to $s + K$. To obtain a simple operator form of the Baxter solution it is now convenient to fix s at the value $s = 0$ in all subsequent considerations. We will also examine the case that $\frac{1}{2}N$ is even, so that henceforth we will consider Baxter families corresponding to $n = 0, 2, 4, \ldots, N$.

The basic idea of this section is to show that the $n \neq 0$ Baxter states arise by the action of simple operators upon the four $n = 0$ states. With $s = 0$, the four $n = 0$ states are explicitly,

$$\psi(1) = (1+k)^{-N/4}\begin{pmatrix}\sqrt{k}\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}-\sqrt{k}\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}\sqrt{k}\\1\end{pmatrix}\otimes\dots$$

$$\psi(2) = (1+k)^{-N/4}\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}-\sqrt{k}\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}\sqrt{k}\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\dots$$

$$\psi(3) = (1+k)^{-N/4}\begin{pmatrix}-\sqrt{k}\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}\sqrt{k}\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}-\sqrt{k}\\1\end{pmatrix}\otimes\dots$$

$$\psi(4) = (1+k)^{-N/4}\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}\sqrt{k}\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}-\sqrt{k}\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\dots. \tag{4.1}$$

First consider the l odd states, $l = 1, 3$. Define, for $x_1 < x_2$, an operator $M(x_1, x_2)$ by

$$M(x_1, x_2) = (-1)^{x_2-x_1-1}(\sigma^+_{x_1}+(-1)^{x_1}\sigma^-_{x_1})[\sigma^z_{x_1+1}\sigma^z_{x_1+2}\dots\sigma^z_{x_2-1}](\sigma^+_{x_2}-(-1)^{x_2}\sigma^-_{x_2}). \tag{4.2}$$

Remembering (2.12) and using

$$\sigma^x_j\varphi_{l_j,l_j+1} = \varphi_{l_j+2,l_j+1}, \qquad \mathrm{i}\sigma^y_j\varphi_{l_j,l_j+1} = \varphi_{l_j,l_j-1}, \tag{4.3}$$

one sees by inspection that for l odd,

$$\psi(l; x_1, x_2) = M(x_1, x_2)\psi(l).$$

Similarly, for l even and $x_1 < x_2$, define

$$N(x_1, x_2) = (-1)^{x_2-x_1-1}(\sigma^+_{x_1}-(-1)^{x_1}\sigma^-_{x_1})[\sigma^z_{x_1+1}\sigma^z_{x_1+2}\dots\sigma^z_{x_2-1}](\sigma^+_{x_2}+(-1)^{x_2}\sigma^-_{x_2}). \tag{4.4}$$

Again one sees that for l even,

$$\psi(l; x_1, x_2) = N(x_1, x_2)\psi(l).$$

Now introduce fermion creation and annihilation operators $c^\dagger_j, c_j$ by the Jordan–Wigner transformation,

$$c_j = \left(\prod_{m=1}^{j-1}\sigma^z_m\right)\sigma^-_j, \qquad c^\dagger_j = \left(\prod_{m=1}^{j-1}\sigma^z_m\right)\sigma^+_j. \tag{4.5}$$

It follows at once that

$$M(x_1, x_2) = (c_{x_1}-(-1)^{x_1}c^\dagger_{x_1})(c_{x_2}-(-1)^{x_2}c^\dagger_{x_2}),$$

$$N(x_1, x_2) = (c_{x_1}+(-1)^{x_1}c^\dagger_{x_1})(c_{x_2}+(-1)^{x_2}c^\dagger_{x_2}). \tag{4.6}$$

This result suggests the introduction of hermitian operators a_j, b_j by

$$\mathrm{i}a_j = \exp(-\tfrac{1}{2}\mathrm{i}\pi j)c_j - \exp(\tfrac{1}{2}\mathrm{i}\pi j)c^\dagger_j,$$

$$b_j = \exp(-\tfrac{1}{2}\mathrm{i}\pi j)c_j + \exp(\tfrac{1}{2}\mathrm{i}\pi j)c^\dagger_j. \tag{4.7}$$

These operators anticommute at different sites,

$$\{a_j, a_{j'}\} = \{a_j, b_{j'}\} = \{b_j, b_{j'}\} = 0, \qquad j \neq j', \tag{4.8}$$

while on the same site

$$\{a_j, b_j\} = 0, \qquad a_j^2 = b_j^2 = 1. \tag{4.9}$$

In terms of a_j, b_j, (4.6) becomes

$$M(x_1, x_2) = \exp\{\tfrac{1}{2}\mathrm{i}\pi(x_1+x_2+2)\}a_{x_1}a_{x_2},$$
$$N(x_1, x_2) = \exp\{\tfrac{1}{2}\mathrm{i}\pi(x_1+x_2)\}b_{x_1}b_{x_2}. \tag{4.10}$$

Now we can express the Baxter states for $n = 2, 4, 6, \ldots, N$ in the following manner. For $l = 1, 3$, and remembering $x_1 < x_2 < \ldots < x_n$,

$$\psi(l; x_1, x_2, \ldots, x_n) = \exp\{\tfrac{1}{2}\mathrm{i}\pi(x_1+x_2+\ldots+x_n+n)\}a_{x_1}a_{x_2}\ldots a_{x_n}\psi(l), \tag{4.11}$$

while for $l = 2, 4$

$$\psi(l; x_1, x_2, \ldots, x_n) = \exp\{\tfrac{1}{2}\mathrm{i}\pi(x_1+x_2+\ldots+x_n)\}b_{x_1}b_{x_2}\ldots b_{x_n}\psi(l). \tag{4.12}$$

These simple operators a_j and b_j when used in pairs serve to turn down pairs of the Baxter 'spins' which are all up in the $n = 0$ states. It is of interest therefore to express $\mathscr{H}_{XY}$ itself in terms of a_j, b_j. First make a decomposition

$$\mathscr{H}_{XY} = \mathscr{H}_{XY}^{(+)}P_+ + \mathscr{H}_{XY}^{(-)}P_-, \tag{4.13}$$

where $\mathscr{H}_{XY}^{(\pm)}$ (after the Jordan–Wigner transformation) is

$$\mathscr{H}_{XY}^{(\pm)} = -\tfrac{1}{2}\sum_{j=1}^{N}\{(c_jc_{j+1}^\dagger - c_j^\dagger c_{j+1}) + k(c_jc_{j+1} - c_j^\dagger c_{j+1}^\dagger)\}. \tag{4.14}$$

In (4.14) one has the convention that for the upper (lower) sign in $\mathscr{H}_{XY}^{(\pm)}$ the c_j satisfy the anticyclic (cyclic) boundary condition $c_{j+N} = \mp c_j$. If we invert (4.7),

$$c_j = \tfrac{1}{2}\exp(\tfrac{1}{2}\mathrm{i}\pi j)(b_j + \mathrm{i}a_j),$$
$$c_j^\dagger = \tfrac{1}{2}\exp(-\tfrac{1}{2}\mathrm{i}\pi j)(b_j - \mathrm{i}a_j), \tag{4.15}$$

we can then obtain

$$\mathscr{H}_{XY}^{(\pm)} = \tfrac{1}{4}\mathrm{i}\sum_{x=1}^{N}[\{1+(-1)^xk\}a_xa_{x+1} + \{1-(-1)^xk\}b_xb_{x+1}], \tag{4.16}$$

where for the upper (lower) sign of $\mathscr{H}_{XY}^{(\pm)}$ the a_x and b_x satisfy the anticyclic (cyclic) boundary condition $a_{x+N} = \mp a_x$, $b_{x+N} = \mp b_x$. From (2.19) one sees that $\{1+(-1)^xk\} = C(x, 0)$. Thus if we denote $C(x, 0)$ by $C(x)$ we have

$$\mathscr{H}_{XY}^{(\pm)} = \tfrac{1}{4}\mathrm{i}\sum_{x=1}^{N}(C(x)a_xa_{x+1} + C(x-1)b_xb_{x+1}). \tag{4.17}$$

5. Operator diagonalization of $\mathscr{H}_{XY}$

We wish now to diagonalize $\mathscr{H}_{XY}^{(\pm)}$ in the form (4.17) by a kind of Fourier transform method. Before doing this, however, let us briefly recall the standard diagonalization (Lieb *et al* 1961) of $\mathscr{H}_{XY}^{(\pm)}$ in the form (4.14). In each subspace $W_\pm$ introduce Fourier transforms of the Jordan–Wigner operators,

$$c_j^\dagger = \frac{1}{\sqrt{N}}\sum_q \exp(-\mathrm{i}qj)\eta_q^\dagger,$$
$$\eta_q^\dagger = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\exp(\mathrm{i}qj)c_j^\dagger, \tag{5.1}$$

where one uses the wavenumbers $q^{\pm}$ of (3.17) and (3.18) in the corresponding subspaces $W_{\pm}$. In terms of the η_q we have

$$\mathscr{H}^{(+)}_{XY} = \sum_{0<q^+<\pi} \{\cos q(\eta_q^\dagger \eta_q + \eta^\dagger_{-q}\eta_{-q}) + \mathrm{i}k \sin q(\eta_q\eta_{-q} + \eta_q^\dagger \eta^\dagger_{-q})\},$$

$$\mathscr{H}^{(-)}_{XY} = \sum_{0<q^-<\pi} \{\cos q(\eta_q^\dagger \eta_q + \eta^\dagger_{-q}\eta_{-q}) + \mathrm{i}k \sin q(\eta_q\eta_{-q} + \eta_q^\dagger \eta^\dagger_{-q})\} + (\eta_0^\dagger\eta_0 - \eta_\pi^\dagger \eta_\pi). \tag{5.2}$$

Make a Bogoliubov–Valatin transformation (Bogoliubov 1958, Valatin 1958) to quasi-particle operators defined by

$$\xi_q^\dagger = \cos(\tfrac{1}{2}\gamma_q)\eta_q^\dagger - \mathrm{i}\sin(\tfrac{1}{2}\gamma_q)\eta_{-q}, \tag{5.3}$$

$$\tan\gamma_q = k\tan q. \tag{5.4}$$

Then (5.2) becomes

$$\mathscr{H}^{(+)}_{XY} = \sum_{0<q^+} E_q(\xi_q^\dagger\xi_q \text{-} \xi^\dagger_{-q}\xi_{-q} - 1),$$

$$\mathscr{H}^{(-)}_{XY} = \sum_{0<q^-<\pi} E_q(\xi_q^\dagger\xi_q \text{-} \xi^\dagger_{-q}\xi_{-q} - 1) + (\xi_0^\dagger\xi_0 \text{-} \xi_\pi^\dagger\xi_\pi - 1), \tag{5.5}$$

with $E_q = (\cos^2 q + k^2 \sin^2 q)^{1/2}$ as in (3.12) with the positive square root taken at all q values.

We would like a similar result for (4.17) and it is easily attained if we note that, when $s = 0$, the factor $\mathrm{i}G(u+s+\frac{1}{2}\mathrm{i}K')$ in (3.14) becomes

$$\mathrm{i}G(u+\tfrac{1}{2}\mathrm{i}K') = \frac{\cos q + \mathrm{i}k\sin q}{E_q} = \exp(\mathrm{i}\gamma_q), \tag{5.6}$$

where γ_q is the angle introduced in (5.3). To prove (5.6) is a simple exercise using the elliptic properties of $G(u)$ together with (3.11) relating q to u and (3.9) expressing E_q in terms of u. At $s = 0$, (3.14) becomes for x odd, $g(x, 0, u) = \mathrm{e}^{\mathrm{i}qx}$, and for x even, $g(x, 0, u) = -\exp(\mathrm{i}\gamma_q)\exp(\mathrm{i}qx)$. Therefore when $s = 0$ we may redefine these single particle wavefunctions by multiplying $g(x, 0, u)$ for all x by the phase factor $\mathrm{i}\exp(-\mathrm{i}\frac{1}{2}\gamma_q)$ to get for x odd,

$$\rho(x, u) = \rho(x, q) = \exp(\tfrac{1}{2}\mathrm{i}\pi)\exp(-\tfrac{1}{2}\mathrm{i}\gamma_q)\exp(\mathrm{i}qx), \tag{5.7}$$

and for x even,

$$\rho(x, u) = \rho(x, q) = \exp(-\tfrac{1}{2}\mathrm{i}\pi)\exp(\tfrac{1}{2}\mathrm{i}\gamma_q)\exp(\mathrm{i}qx). \tag{5.8}$$

In terms of $\rho(x, q)$, equation (3.5) becomes

$$E_q\rho(x, q) = -\tfrac{1}{2}C(x)\rho(x+1, q) - \tfrac{1}{2}C(x-1)\rho(x-1, q), \tag{5.9}$$

where, as in § 4, $C(x)$ denotes $C(x, s)$ at $s = 0$. Now use the single particle wavefunctions $\rho(x, q)$ to define an operator $\alpha_q^\dagger$ by

$$\alpha_q^\dagger = \frac{1}{\sqrt{2N}}\sum_{x=1}^{N} \exp(\tfrac{1}{2}\mathrm{i}\pi x)\rho(x, q)a_x. \tag{5.10}$$

Since the N operators a_x are hermitian, it does not follow that there are N independent $\alpha_q^\dagger$. In fact, from (5.6) it is evident that $\gamma_q = -\gamma_{-q}$ and $\gamma_{q-\pi} = \gamma_q - \pi$, hence $\rho(x, q-\pi) = -\mathrm{i}\rho(x, q)$, and

$$\alpha^\dagger_{q-\pi} = -\mathrm{i}\alpha_q^\dagger. \tag{5.11}$$

Thus in each subspace $W_{\pm}$, as we take the appropriate wavenumbers $q^{\pm}$, we obtain $\frac{1}{2}N$ independent $\alpha_q^\dagger$ together with the corresponding $\frac{1}{2}N$ adjoints. Let us choose the independent set of operators to be those operators with positive wavenumbers, $q^{\pm} > 0$. These $\alpha_q^\dagger$ are proper fermion operators with anticommutation relations

$$\{\alpha_q^\dagger, \alpha_{q'}^\dagger\} = \{\alpha_q, \alpha_{q'}\} = 0,$$
$$\{\alpha_q^\dagger, \alpha_{q'}\} = \delta_{q,q'}. \tag{5.12}$$

These commutation rules are easily verified using

$$\sum_{x=1}^{N} (-1)^x \rho(x, q)\rho(x, q') = 0,$$
$$\sum_{x=1}^{N} \rho(x, q)\rho^*(x, q') = N\delta_{q,q'}, \tag{5.13}$$

where both q and q' are chosen either from the q^+ or from the q^- in (3.17) or (3.18). We have already seen that the operators a_x can be used in pairs to create the $n \neq 0$ Baxter states from the $n = 0$ states when $l = 1, 3$. The operators $\alpha_q^\dagger, \alpha_{q'}$ will do the same thing if applied in pairs to the $n = 0$ states.

Had we chosen the even l states, $l = 2, 4$, at $s = 0$, we would have needed the b_x operators to turn down the Baxter 'spins'. Therefore introduce another set of single particle wavefunctions by

$$\tilde{\rho}(x, q) = \exp(-\tfrac{1}{2}\mathrm{i}\pi)\exp(\tfrac{1}{2}\mathrm{i}\gamma_q)\exp(\mathrm{i}qx), \qquad x \text{ odd},$$
$$\tilde{\rho}(x, q) = \exp(\tfrac{1}{2}\mathrm{i}\pi)\exp(-\tfrac{1}{2}\mathrm{i}\gamma_q)\exp(\mathrm{i}qx), \qquad x \text{ even}. \tag{5.14}$$

These functions satisfy

$$E_q\tilde{\rho}(x, q) = -\tfrac{1}{2}C(x+1)\tilde{\rho}(x+1, q) - \tfrac{1}{2}C(x)\tilde{\rho}(x-1, q). \tag{5.15}$$

We may define another set of operators by

$$\beta_q^\dagger = \frac{1}{\sqrt{2N}} \sum_{x=1}^{N} \exp(\tfrac{1}{2}\mathrm{i}\pi x)\tilde{\rho}(x, q)b_x. \tag{5.16}$$

Again only $\frac{1}{2}N$ of the $\beta_q^\dagger$ are independent since

$$\beta_{q-\pi}^\dagger = \mathrm{i}\beta_q^\dagger. \tag{5.17}$$

The normalization properties (5.13) hold also if ρ is replaced by $\tilde{\rho}$, so that we have anticommutation relations identical to (5.12) for the $\beta_q^\dagger$ and because of (4.8) and (4.9),

$$\{\alpha_q^\dagger, \beta_{q'}^\dagger\} = \{\alpha_q^\dagger, \beta_{q'}\} = \{\alpha_q, \beta_{q'}^\dagger\} = \{\alpha_q, \beta_{q'}\} = 0. \tag{5.18}$$

Both the $\alpha_q^\dagger$ and the $\beta_q^\dagger$ are closely related to the N independent quasiparticle operators $\xi_q^\dagger$. To see this we may use (4.7) and (5.10) together with (5.8) and (5.1) to obtain

$$\alpha_q^\dagger = \frac{1}{\sqrt{2}}\cos\left(\frac{\gamma_q}{2}\right)\eta_q^\dagger - \frac{1}{\sqrt{2}}\cos\left(\frac{\gamma_q}{2}\right)\eta_{\pi-q} - \frac{\mathrm{i}}{\sqrt{2}}\sin\left(\frac{\gamma_q}{2}\right)\eta_{-q} + \frac{\mathrm{i}}{\sqrt{2}}\sin\left(\frac{\gamma_q}{2}\right)\eta_{q-\pi}^\dagger.$$

Remembering that $\gamma_{q-\pi} = \gamma_q - \pi$ and using (5.3) we have

$$\alpha_q^\dagger = \frac{1}{\sqrt{2}}\xi_q^\dagger + \frac{\mathrm{i}}{\sqrt{2}}\xi_{q-\pi}^\dagger. \tag{5.19}$$

Similarly for $\beta_q^\dagger$ we find

$$\beta_q^\dagger = \frac{\mathrm{i}}{\sqrt{2}}\xi_q^\dagger + \frac{1}{\sqrt{2}}\xi_{q-\pi}^\dagger. \tag{5.20}$$

Thus the $\alpha_q^\dagger$, $\beta_q^\dagger$ are simple mixtures of quasiparticle operators arising from a unitary transformation on the degenerate ($E_q = E_{q-\pi}$) pair of operators $\xi_q^\dagger, \xi_{q-\pi}^\dagger$. Now it is trivial to diagonalize $\mathscr{H}_{XY}^{(\pm)}$ in terms of $\alpha_q^\dagger$, $\beta_q^\dagger$ either by substituting in (4.17)

$$a_x = \sqrt{\frac{2}{N}} \sum_{q>0} (\exp(\tfrac{1}{2}\mathrm{i}\pi x)\rho(x,q)\alpha_q + \exp(-\tfrac{1}{2}\mathrm{i}\pi x)\rho^*(x,q)\alpha_q^\dagger),$$

$$b_x = \sqrt{\frac{2}{N}} \sum_{q>0} (\exp(\tfrac{1}{2}\mathrm{i}\pi x)\tilde{\rho}(x,q)\beta_q + \exp(-\tfrac{1}{2}\mathrm{i}\pi x)\tilde{\rho}^*(x,q)\beta_q^\dagger), \tag{5.21}$$

and using (5.9) and (5.15), or by solving for $\xi_q^\dagger$ from (5.19) and (5.20) and then using (5.5). Either way we get a simple result,

$$\mathscr{H}_{XY}^{(\pm)} = \sum_{q^\pm>0} E_q(\alpha_q^\dagger\alpha_q + \beta_q^\dagger\beta_q - 1). \tag{5.22}$$

6. Baxter states in terms of the quasiparticle states

In this section we wish to indicate in some detail the relationship of the quasiparticle states of the standard solution to the Baxter families and to the Bethe-type eigenstates formed from these families. Let us briefly recall the eigenstates of $\mathscr{H}_{XY}$ obtained by the standard quasiparticle diagonalization. In each subspace $W_\pm$ one defines a quasiparticle vacuum state. To do this, introduce the state with all spins turned down,

$$\Phi_{\mathrm{down}} = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}0\\1\end{pmatrix}. \tag{6.1}$$

Then in $W_\pm$ we have vacuum states of the ξ_q operators,

$$\Phi_{\mathrm{vac}}^+ = \prod_{0<q^+<\pi} \{\cos(\tfrac{1}{2}\gamma_q) + \mathrm{i}\sin(\tfrac{1}{2}\gamma_q)\eta_{-q}^\dagger\eta_q^\dagger\}\Phi_{\mathrm{down}}, \tag{6.2}$$

$$\Phi_{\mathrm{vac}}^- = \prod_{0<q^-<\pi} \{\cos(\tfrac{1}{2}\gamma_q) + \mathrm{i}\sin(\tfrac{1}{2}\gamma_q)\eta_{-q}^\dagger\eta_q^\dagger\}(\mathrm{i}\eta_\pi^\dagger)\Phi_{\mathrm{down}}. \tag{6.3}$$

The operator $\eta_\pi^\dagger$ occurs unpaired with any other in (6.3) because from (5.6) $\gamma_\pi = \pi$ and $\xi_\pi^\dagger = -\mathrm{i}\eta_\pi$. Since $P_\mp W_\pm = 0$, in W_+ we have only states corresponding to even numbers of η_q excitations while in W_- we have only states with an odd number of η_q excitations. Remembering the difference between Φ_{vac}^+ and Φ_{vac}^- above, one sees that in $W_\pm$ the eigenstates of $\mathscr{H}_{XY}$ correspond only to those states generated by even numbers of quasiparticle operators acting on the respective vacua. States generated by odd numbers of quasiparticle operators are not eigenstates of $\mathscr{H}_{XY}$ and must be discarded.

We can give an analogous description of the Bethe-type eigenstates of § 3 in terms of operators $\alpha_q^\dagger, \alpha_{q'}$ or $\beta_q^\dagger, \beta_{q'}$ acting upon the $n = 0$ Baxter states. We will discuss in detail how this is done only for the $\alpha_q^\dagger, \alpha_{q'}$ operators acting upon the odd $l(l = 1, 3)$ Baxter states with $n = 0$. The interested reader may easily do a similar calculation for $\beta_q^\dagger, \beta_{q'}$ acting upon even l, $n = 0$ states.

In (4.1) we explicitly wrote the four $n = 0$ states at $s = 0$. We are interested in $\psi(1)$ and $\psi(3)$ which are easily expressed in terms of Φ_{down} by

$$\begin{aligned}\psi(1) &= (1+k)^{-N/4} \prod_{j=1}^{N/2} [1+(-1)^{j-1}\sqrt{k}c^{\dagger}_{2j-1}]\Phi_{\text{down}},\\ \psi(3) &= (1+k)^{-N/4} \prod_{j=1}^{N/2} [1+(-1)^{j}\sqrt{k}c^{\dagger}_{2j-1}]\Phi_{\text{down}},\end{aligned} \tag{6.4}$$

where a definite order of factors is implied in the products. Introduce normalized $n = 0$ states in $W_{\pm}$ by

$$\Psi^{(0)}_{\pm} = N_{\pm}P_{\pm}\psi(1) = \tfrac{1}{2}N_{\pm}(\psi(1) \pm \psi(3)), \tag{6.5}$$

where by (2.17) we may choose

$$N_{\pm} = \sqrt{2}(1+k)^{N/4}\{(1+k)^{N/2} \pm (1-k)^{N/2}\}^{-1/2}. \tag{6.6}$$

Expanding the products in (6.4) gives

$$\begin{aligned}\Psi^{(0)}_{+} &= N_{+}(1+k)^{-N/4} \sum_{\substack{n=0\\(n\,\text{even})}}^{N/2} k^{n/2} \sum_{\substack{j_1<j_2<\ldots<j_n\\(j_i\,\text{odd})}} (-1)^{\frac{1}{2}(j_1+j_2+\ldots+j_n-n)} c^{\dagger}_{j_1}c^{\dagger}_{j_2}\ldots c^{\dagger}_{j_n}\Phi_{\text{down}}\\ \Psi^{(0)}_{-} &= N_{-}(1+k)^{-N/4} \sum_{\substack{n=1\\(n\,\text{odd})}}^{\frac{1}{2}N-1} k^{n/2} \sum_{\substack{j_1<j_2<\ldots<j_n\\(j_i\,\text{odd})}} (-1)^{\frac{1}{2}(j_1+j_2+\ldots+j_n-n)} c^{\dagger}_{j_1}c^{\dagger}_{j_2}\ldots c^{\dagger}_{j_n}\Phi_{\text{down}}.\end{aligned} \tag{6.7}$$

If we define in wavenumber space the operator

$$P_q = \cos\gamma_q + \tfrac{1}{2}\mathrm{i}\sin\gamma_q(\eta^{\dagger}_{-q} - \eta^{\dagger}_{\pi-q})(\eta^{\dagger}_{q} - \eta^{\dagger}_{q-\pi}), \tag{6.8}$$

then as shown in appendix 3 we may express (6.7) as

$$\begin{aligned}\Psi^{(0)}_{+} &= \left(\prod_{0<q^{+}<\pi/2} P_q\right)\Phi_{\text{down}},\\ \Psi^{(0)}_{-} &= \left(\prod_{0<q^{-}<\pi/2} P_q\right)\left(\frac{\mathrm{i}}{\sqrt{2}}\right)(\eta^{\dagger}_{-\pi/2} - \eta^{\dagger}_{\pi/2})\Phi_{\text{down}}.\end{aligned} \tag{6.9}$$

From (6.9) it is straightforward to express $\Psi^{(0)}_{\pm}$ in terms of $\Phi^{\pm}_{\text{vac}}$. Define the operator

$$S_q = \mathrm{i}\{\sin(\tfrac{1}{2}\gamma_q)\xi^{\dagger}_q + \sin(\tfrac{1}{2}\gamma_{q-\pi})\xi^{\dagger}_{q-\pi}\}\{\sin(\tfrac{1}{2}\gamma_{-q})\xi^{\dagger}_{-q} + \sin(\tfrac{1}{2}\gamma_{\pi-q})\xi^{\dagger}_{\pi-q}\}. \tag{6.10}$$

Then one finds

$$\begin{aligned}\Psi^{(0)}_{+} &= \left(\prod_{0<q^{+}<\pi/2} S_q\right)\Phi^{+}_{\text{vac}},\\ \Psi^{(0)}_{-} &= \left(\prod_{0<q^{-}<\pi/2} S_q\right)\left(\frac{\mathrm{i}}{\sqrt{2}}\right)(\xi^{\dagger}_{-\pi/2} - \xi^{\dagger}_{\pi/2})\xi^{\dagger}_{\pi}\Phi^{-}_{\text{vac}}.\end{aligned} \tag{6.11}$$

Conversely one obtains

$$\begin{aligned}\Phi^{+}_{\text{vac}} &= 2^{N/4}\left(\prod_{0<q^{+}<\pi/2} \alpha_q\alpha_{\pi-q}\right)\Psi^{(0)}_{+},\\ \Phi^{-}_{\text{vac}} &= -2^{N/4}\exp(\tfrac{1}{4}\mathrm{i}\pi)\left(\prod_{0<q^{-}<\pi/2} \alpha_q\alpha_{\pi-q}\right)\alpha_{\pi/2}\alpha_{\pi}\Psi^{(0)}_{-}.\end{aligned} \tag{6.12}$$

From (5.19) and (5.20) it is immediate that

$$\alpha_q^\dagger\alpha_q+\beta_q^\dagger\beta_q = \xi_q^\dagger\xi_q+\xi_{q-\pi}^\dagger\xi_{q-\pi},$$

and from (6.10) and (6.11) follows

$$(\alpha_q^\dagger\alpha_q+\beta_q^\dagger\beta_q)\Psi_\pm^{(0)} = \Psi_\pm^{(0)}. \tag{6.13}$$

Looking at (5.22) we see yet again that $\Psi_\pm^{(0)}$ is a zero energy eigenstate of $\mathscr{H}_{XY}^\pm$. However, these $n = 0$ states $\Psi_\pm^{(0)}$ are not eigenstates of the operators $\alpha_q^\dagger\alpha_q$ and $\beta_q^\dagger\beta_q$ separately, nor are they eigenstates of $\xi_q^\dagger\xi_q$. Rather we have the following simple expectation values for these operators:

$$(\Psi_\pm^{(0)}, \alpha_q^\dagger\alpha_q\Psi_\pm^{(0)}) = (\Psi_\pm^{(0)}, \beta_q^\dagger\beta_q\Psi_\pm^{(0)}) = \tfrac{1}{2},$$
$$(\Psi_\pm^{(0)}, \xi_q^\dagger\xi_q\Psi_\pm^{(0)}) = \sin^2(\tfrac{1}{2}\gamma_q), \tag{6.14}$$

where q takes values appropriate to the respective subspaces $W_\pm$.

It is evident from (6.14) that the $n = 0$ states $\Psi_\pm^{(0)}$ are not vacuum states for the operators α_q or $\beta_{q'}$. Nonetheless, in terms of the operators $\alpha_q^\dagger$, $\alpha_{q'}$ and the states $\Psi_\pm^{(0)}$, we can give a complete set of eigenstates of $\mathscr{H}_{XY}$ corresponding, apart from phase, to the Bethe-type eigenstates of § 3 evaluated at $s = 0$. Out of the N distinct operators $\alpha_q^\dagger$, $\alpha_{q'}$ take all possible pairs and then act with 0, 1, 2, ..., $\frac{1}{2}N$ pairs on $\Psi_\pm^{(0)}$. This procedure will produce 2^{N-1} independent states in each subspace and 2^N in the whole space. In doing this one must remember that, since $\Psi_\pm^{(0)}$ is not a vacuum state for α_q, even an operator like $\alpha_q^\dagger\alpha_q$ will produce an independent state when applied to $\Psi_\pm^{(0)}$. In this manner one can give a complete set of states without any reference to $\beta_q^\dagger$ excitations. The Bethe solution state $\Psi_\pm^{(2)}$ of § 3 would at $s = 0$ be simply expressed as

$$\Psi_\pm^{(2)}(q_1, q_2)|_{s=0} = -\exp\{\tfrac{1}{2}\mathrm{i}(\gamma_{q_1}+\gamma_{q_2})\}\alpha_{q_1}^\dagger\alpha_{q_2}^\dagger\Psi_\pm^{(0)},$$

where the phase factor arises from the phase difference between $g(x, 0, u)$ and $\rho(x, u)$. There would be corresponding expressions for other Bethe-type wavefunctions. It is evident that a very similar description of states in terms of $\beta_q^\dagger$ excitations could be given if we had initially used Baxter states with $l = 2, 4$. Such states would have been simply related to the Bethe-type states at $s = K$.

Using the relations above between the quasiparticle vacua and the $n = 0$ Baxter states $\Psi_\pm^{(0)}$ one may now translate completely the Baxter solution for the XY model at $s = 0$ into the quasiparticle language of the standard solution. One of the most interesting features of the Baxter solution as sketched above for the XY model is the fact that the basic states can be taken to describe real 'spins' which are rotated from site to site along the chain. It will be interesting to examine the effect of an external field applied to these 'spin' arrays in the XY model. It will also be of interest to apply the Baxter solution as outlined above to the free-fermion model (Fan and Wu 1969, 1970) which has recently been cast into an elegant operator form in terms of quasiparticle operators (Felderhof 1973). This work, however, along with applications to the XYZ model, we reserve for later publication.

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Appendix 1

Our aim here is to sketch first how one obtains the normalization properties (2.16) and (2.17) for the states $\psi(l; x_1, x_2, \ldots, x_n)$ and secondly how one derives the action of $\mathscr{H}_{j,j+1}$ as given in (2.18). The result (2.16) follows immediately from the definition (2.2) of the up and down Baxter 'spin' states on each site. Consider two states of the same l value, $\psi(l; x_1, x_2, \ldots, x_n)$ and $\psi(l; y_1, y_2, \ldots, y_m)$ which may also be written as

$$\psi(l; x_1, x_2, \ldots, x_n) = \psi(l_1, l_2, l_3, \ldots, l_N, l_{N+1}),$$
$$\psi(l; y_1, y_2, \ldots, y_m) = \psi(l_1, l'_2, l'_3, \ldots, l'_N, l'_{N+1}),$$

where $l_1 = l$ in each case, but the l'_j may differ from the l_j for $j \neq 1$. Since these are direct product states we start with site 1 where φ_{l_1,l_2} and φ_{l_1,l'_2} occur. If one of this pair is an up 'spin' and the other a down 'spin' then they have zero scalar product and $\psi(l; x_1, \ldots, x_n)$ and $\psi(l; y_1, \ldots, y_m)$ are orthogonal. If these two 'spins' are both up or both down then $l_2 = l'_2$ and they contribute a factor one in the scalar product and we move to site two to look at φ_{l_2,l_3} and φ_{l_2,l'_3}. On site two the same possibilities apply and thus we may move from site to site along the chain until we reach the end or reach a site where one 'spin' is up and the other 'spin' is down making $\psi(l; x_1, \ldots, x_n)$ and $\psi(l; y_1, \ldots, y_m)$ orthogonal. We conclude that these two states are orthogonal unless $l_j = l'_j$ for all j in which case $n = m$, $x_1 = y_1$, $x_2 = y_2, \ldots, x_n = y_n$, and we have proved (2.16).

Next consider two states whose l values differ by two, $\psi(l; x_1, \ldots, x_n)$ and $\psi(l+2; y_1, \ldots, y_m)$. Again we may write these as

$$\psi(l; x_1, \ldots, x_n) = \psi(l_1, l_2, \ldots, l_{N+1}),$$
$$\psi(l+2; y_1, \ldots, y_m) = \psi(l'_1, l'_2, \ldots, l'_{N+1}).$$

We first note that these states are orthogonal if $n \neq m$. To see this, assume $m \geqslant n+2$ (if $n \geqslant m+2$ we can shift l to $l+4$ in $\psi(l; x_1, \ldots, x_n)$ and carry out the argument in the same way). If $m \geqslant n+2$, then $l'_1 = l+2 > l_1 = l$, but at the end of the chain $l'_{N+1} = l+2+N-2m < l_{N+1} = l+N-2n$. This is to say, the two sequences $l_1, l_2, \ldots, l_{N+1}$ and $l'_1, l'_2, \ldots, l'_{N+1}$ cross over somewhere along the chain. Let j be the last site on which $l_j = l'_j$. Then $\varphi_{l'_j,l'_{j+1}}$ is a down 'spin' while $\varphi_{l_j,l_{j+1}}$ is an up 'spin' and they are orthogonal making the two product states orthogonal as well.

Assuming $n = m$, we now must show that $\psi(l; x_1, \ldots, x_n)$ and $\psi(l+2; y_1, \ldots, y_n)$ are orthogonal unless $x_i = y_i$ for all i. First suppose $y_1 < x_1$ and write again

$$\psi(l; x_1, x_2, \ldots, x_n) = \psi(l_1, l_2, \ldots, l_{N+1}),$$
$$\psi(l+2; y_1, y_2, \ldots, y_n) = \psi(l'_1, l'_2, \ldots, l'_{N+1}).$$

Then $l_1 = l < l'_1 = l+2$, $l_2 = l+1 < l'_2 = l+3, \ldots$ but $l_{y_1+1} = l+y_1 = l'_{y_1+1}$, that is, the two sequences $l_1, l_2, \ldots, l_{N+1}$ and $l'_1, l'_2, \ldots, l'_{N+1}$ have a common element at site y_1+1. Since $l_{N+1} = l+N-2n < l'_{N+1} = l+2+N-2n$, the sequences must again separate. At the site where the separation occurs there will be an up 'spin' in one state, a down 'spin' in the other state, making the two states orthogonal. If $y_1 > x_1$, write $\psi(l; x_1, \ldots, x_n) = \psi(l+4; x_1, \ldots, x_n)$ and repeat the same argument to conclude that the states are orthogonal unless $y_1 = x_1$. Assuming $y_1 = x_1$ move on to repeat exactly the same argument for $x_2, y_2, \ldots, x_n, y_n$.

Finally, assuming $n = m$, $x_i = y_i$ for all i, we need to calculate $(\psi(l; x_1, \ldots, x_n), \psi(l+2; x_1, \ldots, x_n))$. Referring to (2.2) and recalling $p(l+2, s) = -p(l, s)$ one sees that two up 'spins' at site j contribute a factor

$$(\varphi_{l_j,l_j+1}, \varphi_{l_j+2,l_j+3}) = \frac{1-p^2(l_j, s)}{1+p^2(l_j, s)},$$

while two down 'spins' contribute the same factor.

$$(\varphi_{l_j,l_j-1}, \varphi_{l_j+2,l_j+1}) = (\varphi_{l_j,l_j+1}, \varphi_{l_j+2,l_j+3}).$$

Thus

$$(\psi(l; x_1, \ldots, x_n), \psi(l+2; x_1, \ldots, x_n)) = \prod_{j=1}^{N} \frac{1-p^2(l_j, s)}{1+p^2(l_j, s)}.$$

Remembering (2.8) we see that

$$(\psi(l; x_1, \ldots, x_n), \psi(l+2; x_1, \ldots, x_n)) = \left(\frac{1-p^2(l, s)}{1+p^2(l, s)}\right)^{N/2} \left(\frac{1-p^2(l+1, s)}{1+p^2(l+1, s)}\right)^{N/2}.$$

However it is trivial to show that

$$\frac{1-p^2(l+1, s)}{1+p^2(l+1, s)} = \frac{1-k}{1+k}\left(\frac{1+p^2(l, s)}{1-p^2(l, s)}\right),$$

giving the result (2.17).

To establish (2.18) is not difficult but it is rather tedious in detail. Let us sketch the procedure for (2.18a) where we want to evaluate $\mathscr{H}_{j,j+1}\varphi_{l_j,l_j+1} \otimes \varphi_{l_j+1,l_j+2}$. Since $\mathscr{H}_{j,j+1} = \frac{1}{2}(1+k)\sigma_j^x\sigma_{j+1}^x + \frac{1}{2}(1-k)\sigma_j^y\sigma_{j+1}^y$, and since from (2.2)

$$\sigma^x\varphi_{l,l+1} = \varphi_{l+2,l+1}, \qquad \sigma^x\varphi_{l,l-1} = \varphi_{l-2,l-1},$$
$$\mathrm{i}\sigma^y\varphi_{l,l+1} = \varphi_{l,l-1}, \qquad \mathrm{i}\sigma^y\varphi_{l,l-1} = -\varphi_{l,l+1},$$

we have

$$\mathscr{H}_{j,j+1}\varphi_{l_j,l_j+1}\otimes\varphi_{l_j+1,l_j+2} = \tfrac{1}{2}(1+k)\varphi_{l_j+2,l_j+1} \otimes \varphi_{l_j-1,l_j-2} - \tfrac{1}{2}(1-k)\varphi_{l_j,l_j-1} \otimes \varphi_{l_j+1,l_j}.$$

On sites j and $j+1$ we are working in a four-dimensional subspace of W. An orthonormal basis in this subspace is given by the four product states $\varphi_{l_j,l_j+1} \otimes \varphi_{l_j+1,l_j+2}$, $\varphi_{l_j,l_j+1} \otimes \varphi_{l_j+1,l_j}$, $\varphi_{l_j,l_j-1} \otimes \varphi_{l_j+1,l_j+2}$, and $\varphi_{l_j,l_j-1} \otimes \varphi_{l_j+1,l_j}$. Thus we may certainly write

$$\begin{aligned}\mathscr{H}_{j,j+1}\varphi_{l_j,l_j+1} &\otimes \varphi_{l_j+1,l_j+2} \\ &= D_1\varphi_{l_j,l_j+1} \otimes \varphi_{l_j+1,l_j+2} + D_2\varphi_{l_j,l_j-1} \otimes \varphi_{l_j+1,l_j+2} + D_3\varphi_{l_j,l_j+1} \otimes \varphi_{l_j+1,l_j} \\ &\quad + D_4\varphi_{l_j,l_j-1} \otimes \varphi_{l_j+1,l_j},\end{aligned}$$

where by orthonormality in the two site subspace

$$D_1 = \tfrac{1}{2}(1+k)(\varphi_{l_j,l_j+1}, \varphi_{l_j+2,l_j+1})(\varphi_{l_j+1,l_j+2}, \varphi_{l_j-1,l_j-2}),$$
$$D_2 = \tfrac{1}{2}(1+k)(\varphi_{l_j,l_j-1}, \varphi_{l_j+2,l_j+1})(\varphi_{l_j+1,l_j+2}, \varphi_{l_j-1,l_j-2}),$$
$$D_3 = \tfrac{1}{2}(1+k)(\varphi_{l_j,l_j+1}, \varphi_{l_j+2,l_j+1})(\varphi_{l_j+1,l_j}, \varphi_{l_j-1,l_j-2}),$$
$$D_4 = \tfrac{1}{2}(1+k)(\varphi_{l_j,l_j-1}, \varphi_{l_j+2,l_j+1})(\varphi_{l_j+1,l_j}, \varphi_{l_j-1,l_j-2})-\tfrac{1}{2}(1-k).$$

Evaluating the scalar products by (2.2) one may show that $D_1 = A(l_j, s)$, $D_2 = B(l_j, s)$, $D_3 = -B(l_j+1, s)$, $D_4 = 0$ as in (2.18*a*). The effect of $\mathscr{H}_{j,j+1}$ upon $\varphi_{l_j,l_j-1} \otimes \varphi_{l_j-1,l_j-2}$ in (2.18*d*) is obtained in exactly the same manner.

For (2.18*b*) and (2.18*c*) one proceeds in two steps. First one shows

$$\begin{aligned}\mathscr{H}_{j,j+1}\varphi_{l_j,l_j+1} \otimes \varphi_{l_j+1,l_j} &= -A(l_j, s)\varphi_{l_j,l_j+1} \otimes \varphi_{l_j+1,l_j}+(1-k)\varphi_{l_j,l_j-1} \otimes \varphi_{l_j+1,l_j+2} \\ &\quad -B(l_j, s)\varphi_{l_j,l_j-1} \otimes \varphi_{l_j+1,l_j}-B(l_j+1, s)\varphi_{l_j,l_j+1} \otimes \varphi_{l_j+1,l_j+2}\end{aligned}$$

and

$$\begin{aligned}\mathscr{H}_{j,j+1}\varphi_{l_j,l_j-1} \otimes \varphi_{l_j-1,l_j} &= -A(l_j-1, s)\varphi_{l_j,l_j-1} \otimes \varphi_{l_j-1,l_j}+(1-k)\varphi_{l_j,l_j+1} \otimes \varphi_{l_j-1,l_j-2} \\ &\quad -B(l_j, s)\varphi_{l_j,l_j+1} \otimes \varphi_{l_j-1,l_j}-B(l_j-1, s)\varphi_{l_j,l_j-1} \otimes \varphi_{l_j-1,l_j-2}\end{aligned}$$

in exactly the same way as above. Then the identities

$$(1-k)\varphi_{l-1,l-2}-2B(l, s)\varphi_{l-1,l} = C(l, s)\varphi_{l+1,l},$$

and

$$(1-k)\varphi_{l+1,l+2}-2B(l, s)\varphi_{l+1,l} = C(l, s)\varphi_{l-1,l},$$

lead at once to (2.18*b*) and (2.18*c*).

Appendix 2

Let us first give a proof that (3.8) and (3.9) constitute a solution to the recurrence relation (3.7) and then let us examine the function $G(u)$ of (3.10) to see what values of u satisfy the boundary condition (3.16). As a function of u, the elliptic function $G(u)$ is doubly periodic with periods $2K$, $2iK'$ and is of order two. From (3.10*a*) it is trivial to verify the following properties of $G(u)$:

$$G(u) = -G(-u), \qquad G(u+K) = -1/G(u), \qquad G(u+iK') = -G(u). \tag{A.1}$$

We want to establish that the functions $E(u) = 1/\mathrm{sn}(2u)$ and $f(x, s, u) = iG(u)/G(u+\tfrac{1}{2}iK'+s+xK)$ of (3.8) and (3.9) satisfy the relation (3.7),

$$E(u) = A(x-1, s)-\tfrac{1}{2}C(x, s)f(x, s, u)-\tfrac{1}{2}C(x-1, s)/f(x-1, s, u). \tag{A.2}$$

From (A.1) we have that

$$1/f(x-1, s, u) = f(x, s, u)/G^2(u),$$

and using the definition of $A(x, s)$, $C(x, s)$ and simple elliptic properties, we can express (A.2) as

$$E(u) = -k\,\mathrm{sn}(2s+2xK)+R(x, s, u) \tag{A.3}$$

with

$$R(x,s,u) = -\mathrm{i}\left\{(1+k)\left(\frac{1-p^2(x,s)}{1+p^2(x,s)}\right)G(u)+(1-k)\left(\frac{1+p^2(x,s)}{1-p^2(x,s)}\right)\frac{1}{G(u)}\right\}$$
$$\times\{2G(u+\tfrac{1}{2}\mathrm{i}K'+s+xK)\}^{-1}.$$

Now examine the two sides of (A.3) with x, s held fixed and consider the u dependence. The function $E(u)$ is doubly periodic in u with periods $2K$, $\mathrm{i}K'$ and it has two poles within its fundamental domain at $u = 0$ and $u = K$ with residues $\frac{1}{2}$, $-\frac{1}{2}$ respectively. It is easy to see that $R(x, s, u)$ has exactly the same properties with respect to u if we use the identity

$$G(u+\tfrac{1}{2}\mathrm{i}K') = -\mathrm{i}\sqrt{\frac{1+k}{1-k}}\left(\frac{1-k\,\mathrm{sn}^2(u)}{1+k\,\mathrm{sn}^2(u)}\right)$$

to help evaluate the residues of $R(x, s, u)$. Therefore the difference $E(u)-R(x,s,u)$ is, with respect to u, an elliptic function with no poles. By the Liouville theorem (Whittaker and Watson 1965) we conclude that $E(u)-R(x,s,u)$ is a function only of x and s; by putting $u = \frac{1}{2}\mathrm{i}K'$ one easily sees that $E(u)-R(x,s,u)$ is equal to $-k\,\mathrm{sn}(2s+2xK)$.

Because of the boundary condition (3.16) we would like to determine as far as possible the values of $u^{\pm} = v^{\pm}-\frac{1}{2}K$ which satisfy

$$G(v^{\pm}-\tfrac{1}{2}K) = \frac{H(v^{\pm}+\frac{1}{2}K)\Theta(v^{\pm}+\frac{1}{2}K)}{H(v^{\pm}-\frac{1}{2}K)\Theta(v^{\pm}-\frac{1}{2}K)} = \exp(\mathrm{i}q^{\pm}) = (\mp 1)^{1/N}. \tag{A.4}$$

Since $G(u)$ is of order two it will take each such value (A.4) twice within its fundamental region defined by the rectangle with corners $u = -K-\mathrm{i}K'$, $K-\mathrm{i}K'$, $K+\mathrm{i}K'$, $-K+\mathrm{i}K'$. More generally than (A.4) we may ask for what values of $u = v-\frac{1}{2}K$ in this fundamental domain do we have

$$\left|\frac{H(v+\frac{1}{2}K)\Theta(v+\frac{1}{2}K)}{H(v-\frac{1}{2}K)\Theta(v-\frac{1}{2}K)}\right| = 1. \tag{A.5}$$

From (A.1) we see that if u has the values $u = -\frac{1}{2}K+\mathrm{i}\delta$, ($\delta$ real), then

$$G(-\tfrac{1}{2}K+\mathrm{i}\delta) = -1/G(\tfrac{1}{2}K+\mathrm{i}\delta) = 1/G(-\tfrac{1}{2}K-\mathrm{i}\delta) = 1/G^*(-\tfrac{1}{2}K+\mathrm{i}\delta),$$

and hence $|G(u)| = 1$ when $u = -\frac{1}{2}K+\mathrm{i}\delta$. Equation (A.1) at once shows that $|G(u)| = 1$ also when $u = \frac{1}{2}K+\mathrm{i}\delta$. In terms of v, (A.5) is satisfied along the two lines $v = \mathrm{i}\delta$, $v = K+\mathrm{i}\delta$. From (A.5) we may learn more. For we may use the expressions (Whittaker and Watson 1965)

$$\frac{\Theta(v+\eta)}{\Theta(v-\eta)} = \exp(2vZ(\eta)-2\Pi(v,\eta)),$$
$$\frac{H(v+\eta)}{H(v-\eta)} = \exp\left(\mathrm{i}\pi\frac{\eta}{K}+2(v+\mathrm{i}K')Z(\eta)-2\Pi(v+\mathrm{i}K',\eta)\right), \tag{A.6}$$

where $Z(\eta)$ is Jacobi's zeta-function and $\Pi(v,\eta)$ is the elliptic integral of the third kind (Whittaker and Watson 1965). When $v = \mathrm{i}\delta$, then

$$\Pi(\mathrm{i}\delta,\tfrac{1}{2}K) = -\mathrm{i}\left(\frac{k'k^2}{1+k'}\right)\int_0^{\delta}\frac{\mathrm{sn}^2(\delta,k')\,\mathrm{d}\delta}{\mathrm{cn}^2(\delta,k')+k^2\,\mathrm{sn}^2(\frac{1}{2}K,k)\,\mathrm{sn}^2(\delta,k')}$$

is imaginary and, since $Z(\frac{1}{2}K)$ is real, the entire argument of each of the exponentials in (A.6) is imaginary. Further, we observe that as δ increases,

$$\arg\left(\frac{H(v+\frac{1}{2}K)\Theta(v+\frac{1}{2}K)}{H(v-\frac{1}{2}K)\Theta(v-\frac{1}{2}K)}\right)$$

increases monotonically.

From these relations we may conclude that when $u = -\frac{1}{2}K+\mathrm{i}\delta$, $-K' \leqslant \delta < K'$, then $|G(u)| = 1$ and the wavenumber q, which is the argument of $G(u)$, increases from zero at $u = -\frac{1}{2}K-\mathrm{i}K'$, to π at $u = -\frac{1}{2}K$, to 2π at $u = -\frac{1}{2}K+\mathrm{i}K'$. For the choice $u = \frac{1}{2}K+\mathrm{i}\delta$, $-K' \leqslant \delta < K'$, again $|G(u)| = 1$ and q decreases from π at $u = \frac{1}{2}K-\mathrm{i}K'$, to zero at $u = \frac{1}{2}K$, and to $-\pi$ at $u = \frac{1}{2}K+\mathrm{i}K'$. Since $G(u)$ is of order two, these two line segments contain all solutions to (A.4). The energy $E(u)$ is given along these two line segments by

$$E(u) = 1/\mathrm{sn}(2u, k) = 1/\mathrm{sn}(\mp K+\mathrm{i}2\delta, k) = \mp\mathrm{dn}(2\delta, k'). \tag{A.7}$$

In § 3 we wished to choose the positive square root $E_q = (\cos^2 q+k^2\sin^2 q)^{1/2}$, which means that we must choose solutions of (A.4) along the line segment $u = \frac{1}{2}K+\mathrm{i}\delta$, $-K' \leqslant \delta < K'$. Once we specify this choice, then all u dependent quantities are uniquely determined.

Appendix 3

Here we will indicate briefly how to obtain the form (6.9) for the Baxter states $\Psi_{\pm}^{(0)}$ from the expressions given in (6.7). First consider $\Psi_{+}^{(0)}$ given in (6.7) as

$$\Psi_{+}^{(0)} = N_{+}(1+k)^{-N/4}\sum_{\substack{n=0\\(n\text{ even})}}^{N/2} k^{n/2}\sum_{\substack{j_1<j_2<\ldots<j_n\\(j_i\text{ odd})}}(-1)^{\frac{1}{2}(j_1+j_2+\ldots+j_n-n)}c_{j_1}^{\dagger}c_{j_2}^{\dagger}\ldots c_{j_n}^{\dagger}\Phi_{\text{down}}.$$

We proceed in two steps. First we establish that

$$\sum_{\substack{n=0\\(n\text{ even})}}^{N/2} k^{n/2}\sum_{\substack{j_1<j_2<\ldots<j_n\\(j_i\text{ odd})}}(-1)^{\frac{1}{2}(j_1+j_2+\ldots+j_n-n)}c_{j_1}^{\dagger}c_{j_2}^{\dagger}\ldots c_{j_n}^{\dagger}$$
$$= \exp\left(k\sum_{\substack{j_1<j_2\\(j_i\text{ odd})}}(-1)^{\frac{1}{2}(j_1+j_2-2)}c_{j_1}^{\dagger}c_{j_2}^{\dagger}\right). \tag{A.8}$$

It is evident that the first two terms on each side of (A.8) are the same. For terms of higher order in k we may use a simple inductive argument. First assume that

$$\left(k\sum_{\substack{j_1<j_2\\(j_i\text{ odd})}}(-1)^{\frac{1}{2}(j_1+j_2-2)}c_{j_1}^{\dagger}c_{j_2}^{\dagger}\right)^{r-1}$$
$$= (r-1)!\sum_{\substack{j_1<j_2<\ldots<j_{2r-2}\\(j_i\text{ odd})}}(-1)^{\frac{1}{2}(j_1+j_2+\ldots+j_{2r-2}-2r+2)}c_{j_1}^{\dagger}c_{j_2}^{\dagger}\ldots c_{j_{2r-2}}^{\dagger}. \tag{A.9}$$

In order k^r we should then obtain a result like (A.9) but with the term

$$k^r(r!)\sum_{\substack{j_1<j_2<\ldots<j_{2r}\\(j_i\text{ odd})}}(-1)^{\frac{1}{2}(j_1+j_2+\ldots+j_{2r}-2r)}c_{j_1}^{\dagger}c_{j_2}^{\dagger}\ldots c_{j_{2r}}^{\dagger}$$

on the right-hand side. Consider a particular term, say $c_{t_1}^\dagger c_{t_2}^\dagger \ldots c_{t_{2r}}^\dagger$ where $t_1 < t_2 < \ldots < t_{2r}$. Such a term can arise from the product of

$$\left(k \sum_{\substack{j_1<j_2 \\ (j_i \text{ odd})}} (-1)^{\frac{1}{2}(j_1+j_2-2)} c_{j_1}^\dagger c_{j_2}^\dagger\right)$$

with the right-hand side of (A.9) in as many ways as we can choose two operators from among the $2r$ operators $c_{t_i}^\dagger$, that is, in $\binom{2r}{2} = r(2r-1)$ ways. However, some of these terms come with a positive sign and others with a negative sign because of the operator re-ordering required to reach the standard order $t_1 < t_2 < \ldots < t_{2r}$. There will be a positive (negative) sign depending on whether an even (odd) number of factors separate the two chosen operators in $c_{t_1}^\dagger c_{t_2}^\dagger \ldots c_{t_{2r}}^\dagger$. We may choose pairs separated by an even number of factors in $(2r-1)+(2r-3)+\ldots+3+1$ ways and pairs separated by an odd number of factors in $(2r-2)+(2r-4)+\ldots+2$ ways. Therefore in the product we will obtain $c_{t_1}^\dagger c_{t_2}^\dagger \ldots c_{t_{2r}}^\dagger$ exactly r times which completes the induction.

Next consider the operator defined in (6.8),

$$P_q = \cos\gamma_q + \tfrac{1}{2}\mathrm{i}\sin\gamma_q(\eta_{-q}^\dagger - \eta_{\pi-q}^\dagger)(\eta_q^\dagger - \eta_{q-\pi}^\dagger),$$

and the normalized state

$$\Phi = \left(\prod_{0<q^+<\pi/2} P_q\right)\Phi_{\text{down}}. \tag{A.10}$$

The state Φ may also be written as

$$\Phi = \left(\prod_{0<q^+<\pi/2} \cos\gamma_q\right) \exp\left(\tfrac{1}{2}\mathrm{i} \sum_{0<q^+<\pi/2} \tan\gamma_q(\eta_{-q}^\dagger - \eta_{\pi-q}^\dagger)(\eta_q^\dagger - \eta_{q-\pi}^\dagger)\right)\Phi_{\text{down}}.$$

However, $\tan\gamma_q = k\tan q$, and using (5.1) we obtain

$$\Phi = \left(\prod_{0<q^+<\pi/2} \cos\gamma_q\right) \exp\left\{-\frac{4k}{N} \sum_{\substack{j_1<j_2 \\ (j_i \text{ odd})}} \left(\sum_{0<q^+<\pi/2} \tan q \sin q(j_2-j_1)\right) c_{j_1}^\dagger c_{j_2}^\dagger\right\}\Phi_{\text{down}}. \tag{A.11}$$

It is easy to show that for j_1, j_2 odd, and $0 < j_2 - j_1 < N$,

$$-\frac{4}{N}\sum_{0<q^+<\pi/2} \tan q \sin q(j_2-j_1) = (-1)^{\frac{1}{2}(j_2+j_1-2)}. \tag{A.12}$$

Combining (A.11) and (A.12) and then comparing with (A.8) shows that

$$\Psi_+^{(0)} = \Phi = \left(\prod_{0<q^+<\pi/2} P_q\right)\Phi_{\text{down}}.$$

The derivation for $\Psi_-^{(0)}$ is rather similar but with a few changes. Instead of (A.8) and (A.12) one now needs

$$\begin{aligned}\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\frac{1}{2}N-1} k^{n/2} & \sum_{\substack{j_1<j_2<\ldots<j_n \\ (j_i \text{ odd})}} (-1)^{\frac{1}{2}(j_1+j_2+\ldots+j_n-n)} c_{j_1}^\dagger c_{j_2}^\dagger \ldots c_{j_n}^\dagger \\ &= \left(k^{1/2}\sum_{j \text{ odd}} (-1)^{\frac{1}{2}(j-1)} c_j^\dagger\right) \exp\left(-\frac{4k}{N}\sum_{\substack{j_1<j_2 \\ (j_i \text{ odd})}} (-1)^{\frac{1}{2}(j_1+j_2-2)}\right. \\ &\quad \left.\times\{\tfrac{1}{2}(j_2-j_1) - \tfrac{1}{4}N\} c_{j_1}^\dagger c_{j_2}^\dagger\right),\end{aligned} \tag{A.13}$$

and

$$\sum_{0<q^{-}<\pi/2} \tan q \sin q(j_2-j_1) = (-1)^{\frac{1}{2}(j_1+j_2-2)}\{\tfrac{1}{2}(j_2-j_1)-\tfrac{1}{4}N\}. \tag{A.14}$$

These results can be easily established and lead to the result for $\Psi_{-}^{(0)}$ quoted in (6.9).

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